BAKER–CAMPBELL–HAUSDORFF RELATIONS FOR FINITE-DIMENSIONAL LIE ALGEBRAS*

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Abstract

A method is proposed for disentangling exponentials of operators that belong to finite-dimensional Lie algebras. A straightforward matrix representation is combined with a widely used parameter-differentiation method, giving rise to a simpler and more systematic procedure. The SU(1, 1), SU(2) and double photon algebras are explicitly considered as illustrative examples.

1. Introduction

Ordered products of exponential operators are found to be useful in many physical applications [1–6]. For this reason, the Baker–Campbell–Hausdorff (BCH) [7] relations have been widely studied. Among the methods proposed to obtain them, we mention the use of similarity transformations [8], coherent states [9], finite matrix representations [1,6,7,10,11] and several forms of parameter differentiation [12–15]. Particularly interesting is the case of exponentials of operators belonging to a finite-dimensional Lie algebra [1–6,8–13,15].

The parameter-differentiation method (PDM) [13–15] is powerful and straightforward, and has recently been applied to supergroups [14]. It reduces the problem of obtaining the BCH relations to solving a system of first-order, ordinary nonlinear differential equations [13–15]. However, the PDM does not seem to be systematic enough because a general method for integrating the above-mentioned differential equations has not been given.

For this reason, a matrix representation, which is closely related to the adjoint one [16], is proposed here to obtain most (if not all) of the solutions of the PDM differential equations in a straightforward, systematic way. This matrix representation has recently been applied to the solution of the Schrödinger equation for time-dependent, quantum-mechanical models [17], to the calculations of transition

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probabilities [18] and other matrix elements [19], and to the construction of exponential time-evolution operators [20] which prove to be useful in studying the convergence properties of the Magnus expansion [21]. The global validity of the time-evolution operator in the ordered product form has also been discussed in this way [22]. One of the main advantages of this matrix representation with regard to the treatment of time-dependent models is that it enables one to avoid the use of a particular form for the time-evolution operator, thus overcoming the serious problem of the lack of globality of such expressions [22,23].

In section 2, the PDM is briefly reviewed and the matrix representation is introduced. The SU(1, 1), SU(2) and double photon algebras are treated in section 3 as particular examples. Further comments and conclusions are found in section 4.

2. The method

Since the properties of the finite-dimensional Lie algebras are well known in many field of physics [16,24], we do not deem it necessary to summarize them here. A set of operators $\{X_1, \ldots, X_n\}$ is said to span a finite-dimensional Lie algebra $\mathcal{L}_X$ if

$$[X_j, X_k] = X_j X_k - X_k X_j = \sum_{m=1}^{n} C_{jk}^m X_m, \quad j,k = 1,2,\ldots,n, \tag{1}$$

where the complex numbers C_{jk}^m are called structure constants. Every element A of $\mathcal{L}_X$ can be written

$$A = \sum_{k=1}^{n} a_k X_k, \tag{2}$$

where the coefficients a_k are complex numbers. We concentrate on the problem of rewriting a given exponential operator

$$O = \exp\left[\sum_{k=1}^{n} \alpha_k X_k\right] = O_{\mathrm{I}}(\boldsymbol{\alpha}), \tag{3}$$

in the ordered product form

$$O = \prod_{k=1}^{n} \exp(\beta_k X_k) = O_{\mathrm{II}}(\boldsymbol{\beta}), \tag{4}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are column matrices with complex elements α_j and β_j, $j = 1, 2, \ldots, n$, respectively. The parameters α_j and β_j are called canonical coordinates of the first and second kind, respectively [13], and the relation between them is a particular case of the BCH formulas.

In order to obtain $\boldsymbol{\beta}$ in terms of $\boldsymbol{\alpha}$, the PDM [13–15] has been proposed which consists of defining

$$O(\lambda) = O_{\mathrm{I}}(\lambda\boldsymbol{\alpha}) = \exp\left[\lambda \sum_{k=1}^{n} \alpha_k X_k\right], \tag{5}$$

so that $O(0) = I$ is the identity operator and $O(1)$ is the exponential operator (3). The method is based on the assumption that the canonical coordinates of the second kind $\boldsymbol{\beta}(\lambda)$, obtained from $O_{\mathrm{II}}[\boldsymbol{\beta}(\lambda)] = O_{\mathrm{I}}(\lambda\boldsymbol{\alpha})$, are differentiable functions of λ for all $0 < \lambda < 1$. In such a case, it follows that

$$\dot{O}_{\mathrm{II}}[\beta(\lambda)]O_{\mathrm{II}}^{-1}[\beta(\lambda)] = \dot{O}_{\mathrm{I}}(\lambda\boldsymbol{\alpha})O_{\mathrm{I}}^{-1}(\lambda\boldsymbol{\alpha}) = \sum_{k=1}^{n} \alpha_k X_k, \tag{6}$$

where the dot stands for differentiation with respect to λ. A straightforward algebraic manipulation shows that $\boldsymbol{\alpha} = \mathbf{M}\,\dot{\boldsymbol{\beta}}$, where the elements and the $n \times n$ matrix $\mathbf{M}$ are analytical functions of $\boldsymbol{\beta}(\lambda)$. If this matrix is non-singular for all $0 < \lambda < 1$, then the first-order nonlinear equations

$$\dot{\boldsymbol{\beta}} = \mathbf{M}^{-1}\boldsymbol{\alpha}, \quad \boldsymbol{\beta}(0) = \boldsymbol{0}, \tag{7}$$

can be integrated, and the BCH relations are merely given by $\boldsymbol{\beta}(1)$. However, it seems that a good deal of ingenuity is necessary to solve eq. (7), which most commonly looks rather intricate.

For this reason, in what follows we develop a straightforward, systematic way of obtaining the BCH formulas which, as far as we know, has not been used before although the main underlying ideas may be known. Let $\mathcal{L}_Y$ be an operator space spanned by $\{Y_1, Y_2, \ldots, Y_m\}$, $m \leq n$, that satisfies $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_Y$. Notice that there is always at least one such space because $\mathcal{L}_X$ satisfies the definition above. (In this case, the representation is called adjoint or regular [16].) If more than one space $\mathcal{L}_Y$ is found, the one with the smallest dimension is chosen.

Every operator A in $\mathcal{L}_X$ can be related to an $m \times m$ complex matrix $\mathbf{A} = (\mathbf{A}_{jk})$ as follows:

$$[A, Y_j] = \sum_{k=1}^{m} \mathbf{A}_{kj} Y_k, \qquad j = 1, 2, \ldots, m. \tag{8}$$

In addition, for the exponential operator $U = \mathrm{e}^A$, one has

$$U Y_j U^{-1} = \sum_{k=1}^{m} \mathbf{U}_{kj} Y_k, \quad j = 1, \ldots, m, \tag{9}$$

where

$$\mathbf{U} = \mathrm{e}^{\mathbf{A}}. \tag{10}$$

Equations (9) and (10) follow from the well-known expansion

$$e^{A}B\,e^{-A} = B + [A,B] + [A,[A,B]]/2 + \dots . \qquad (11)$$

Given the structure constants C_{jk}^{m} and the form of A, the calculation of $\mathbf{A}$ offers no difficulty. The matrix $\mathbf{U}$ is easily obtained by means of well-known methods of the linear algebra [25]. In addition, if U_1 and U_2 are two exponential operators with matrices $\mathbf{U}_1$ and $\mathbf{U}_2$, respectively, then the product operator U_1U_2 is represented by the matrix $\mathbf{U}_1\mathbf{U}_2$. For this reason, it is not difficult to obtain the matrices $\mathbf{O}_{\mathrm{I}}(\lambda\boldsymbol{\alpha})$ and $\mathbf{O}_{\mathrm{II}}[\boldsymbol{\beta}(\lambda)]$ corresponding to $O_{\mathrm{I}}(\lambda\boldsymbol{\alpha})$ and $O_{\mathrm{II}}[\boldsymbol{\beta}(\lambda)]$, respectively. As a result, some of the BCH relations, if not all of them, follow from the matrix equality

$$\mathbf{O}_{\mathrm{II}}[\boldsymbol{\beta}(\lambda)] = \mathbf{O}_{\mathrm{I}}(\lambda\boldsymbol{\alpha}), \qquad (12)$$

which is an obvious consequence of the fact that if $O_{\mathrm{I}} = O_{\mathrm{II}}$, then $O_{\mathrm{I}}Y_jO_{\mathrm{I}}^{-1} = O_{\mathrm{II}}Y_jO_{\mathrm{II}}^{-1}$ for all $j = 1, 2, \dots, m$. The calculation of the matrices for all the exponential operators in eq. (12) is not lengthy, as it may appear at first sight, because they can be easily derived from a general exponential $e^{\mathbf{A}}$, where A is an arbitrary element of $\mathcal{L}_X$ such as the one in eq. (2).

In some cases, not all the BCH formulas can be obtained in this way because the matrix representation just discussed in not faithful [16]. For instance, if one of the operators in $\mathcal{L}_X$ (say X_1) commutes with all the ones in $\mathcal{L}_Y$, then all the operators of the form $A + \gamma X_1$ are related to the same matrix $\mathbf{A}$. Since in such a case the matrix $\mathbf{O}_{\mathrm{II}}$ is independent of β_1, this parameter cannot be obtained from eq. (12). However, it can certainly be calculated by means of the PDM which now reduces to just one nonlinear first-order ordinary differential equation because $\beta_2(\lambda), \beta_3(\lambda), \dots, \beta_n(\lambda)$ are obtained from eq. (12). To see this more clearly, notice that the first row of $\boldsymbol{\alpha} = \mathbf{M}\,\dot{\boldsymbol{\beta}}$ can be written $\dot{\beta}_1(\lambda) = \alpha_1 - \mathbf{M}_{12}\dot{\beta}_2(\lambda) - \mathbf{M}_{13}\dot{\beta}_3(\lambda) - \dots - \mathbf{M}_{1n}\dot{\beta}_n(\lambda)$ because $\mathbf{M}_{11} = 1$. It is worth pointing out that $\mathbf{M}^{-1}$ is not required to obtain the remaining parameter $\beta_1(l)$ by quadratures.

Faithful matrix representations for most Lie algebras with applications in physics are available [1,10,11,16]. However, in order to make use of them a good deal of familiarity with Lie algebras and groups is required. On the other hand, the method above is based on elementary operator algebra, and for this reason it is proposed in this paper.

3. Examples

The matrix representation method discussed in the previous section is useful in treating many physical problems. Three examples of disentangling exponential operators are considered in what follows.

3.1. DOUBLE PHOTON ALGEBRA

To begin with, consider the algebra spanned by $\{X_1 = I, X_2 = a^+, X_3 = (a^+)^2, X_4 = a^+a, X_5 = a, X_6 = a^2\}$, where

$$[a, a^+] = I. \tag{13}$$

When a^+ is the adjoint of a, the well-known boson algebra is obtained. However, such a condition is not invoked here so that the results are as general as possible. It is worth noticing that the basis order has been chosen in such a way that the product form (4) in the case of the boson algebra corresponds to the normal ordering provided that $\exp(\beta_4 a^+ a)$ is written in normal ordered form [26].

A straightforward calculation shows that eq. (7) becomes

$$\begin{aligned}
\dot\beta_1 &= \alpha_1 + \beta_2\alpha_5 + (\beta_2^2 + 2\beta_3)\alpha_6 , \\
\dot\beta_2 &= \alpha_2 + 2\beta_3\alpha_5 + 4\beta_2\beta_3\alpha_6 + \beta_2\alpha_4 , \\
\dot\beta_3 &= \alpha_3 + 4\beta_3^2\alpha_6 + 2\beta_3\alpha_4 , \\
\dot\beta_4 &= \alpha_4 + 4\beta_3\alpha_6 , \\
\dot\beta_5 &= \exp(\beta_4)\alpha_5 + 2\beta_2\exp(\beta_4)\alpha_6 , \\
\dot\beta_6 &= \exp(2\beta_4)\alpha_6 .
\end{aligned} \tag{14}$$

In order to obtain the BCH formulas, one has to integrate eqs. (14) with the initial conditions $\boldsymbol{\beta}(0) = \boldsymbol{0}$. Although the calculation does not seem to be straightforward, similar equations have certainly been solved for the superalgebra osp(1/2) [14].

The application of the method proposed in this paper is remarkably simple if $\mathcal{L}_Y$ is chosen to be the set of operators spanned by $\{Y_1 = I, Y_2 = a^+, Y_3 = a\}$ because a tractable 3×3 matrix representation is obtained (the faithful matrix representation for this algebra is fourth-dimensional [6,16]). In this case, $\mathcal{L}_Y$ is an ideal of $\mathcal{L}_X$ [16,24]. According to eq. (8), the matrix for a general operator A of the form (2) belonging to $\mathcal{L}_X$ is

$$\mathbf{A} = \begin{pmatrix} 0 & a_5 & -a_2 \\ 0 & a_4 & -2a_3 \\ 0 & 2a_6 & -a_4 \end{pmatrix}, \tag{15}$$

from which one can easily derive the matrix representation for the basis operator X_j.

The parameters $\beta_2(\lambda), \beta_3(\lambda), \ldots, \beta_6(\lambda)$ are obtained from eq. (12), where the matrices $\mathbf{O}_{\mathrm{II}}[\boldsymbol{\beta}(\lambda)]$ and $\mathbf{O}_{\mathrm{I}}(\lambda\boldsymbol{\alpha})$ are, respectively, given by

$$\begin{pmatrix} 1 & \beta_5 - 2\beta_2\beta_6 \exp(-\beta_4) & -\beta_2 \exp(-\beta_4) \\ 0 & -4\beta_3\beta_6 \exp(-\beta_4) + \exp(\beta_4) & -2\beta_3 \exp(-\beta_4) \\ 0 & 2\beta_6 \exp(-\beta_4) & \exp(-\beta_4) \end{pmatrix}, \tag{16}$$

and

$$\begin{pmatrix} 1 & \begin{matrix}\delta_2[1-\cos(\lambda\delta)]/\delta^2 \\ +\alpha_5 \sin(\lambda\delta)/\delta\end{matrix} & \begin{matrix}\delta_1[1-\cos(\lambda\delta)]/\delta^2 \\ -\alpha_2 \sin(\lambda\delta)/\delta\end{matrix} \\ 0 & \cos(\lambda\delta) + \alpha_4 \sin(\lambda\delta)/\delta & -2\alpha_3 \sin(\lambda\delta)/\delta \\ 0 & 2\alpha_6 \sin(\lambda\delta)/\delta & \cos(\lambda\delta) - \alpha_4 \sin(\lambda\delta)/\delta \end{pmatrix}, \tag{17}$$

with $\delta^2 = 4\alpha_3\alpha_6 - \alpha_4^2$, $\delta_1 = \alpha_2\alpha_4 - 2\alpha_3\alpha_5$ and $\delta_2 = \alpha_5\alpha_4 - 2\alpha_2\alpha_6$. One can easily derive an alternative equivalent expression for the latter matrix by substituting $-\Delta^2$, $\cosh(\lambda\Delta)$ and $\sinh(\lambda\Delta)/\Delta$ for δ^2, $\cos(\lambda\delta)$ and $\sin(\lambda\delta)/\delta$, respectively.

As argued before, β_1 cannot be obtained in this way because the matrix representation is not faithful. This parameter can be easily obtained as discussed in the previous section. However, here we use the first equation in (14), which leads to

$$\beta_1(\lambda) = \alpha_1\lambda + \int_0^\lambda \{\alpha_5\beta_2(\lambda') + \alpha_6[\beta_2(\lambda')^2 + 2\beta_3(\lambda')]\}\, d\lambda'. \tag{18}$$

Finally, to obtain the BCH formulas, it is only necessary to set $\lambda = 1$.

The double photon algebra is useful in treating time-dependent harmonic oscillators [1,4–6]. If the coordinate–momentum representation is used, the matrix method developed above leads to a well-known representation for the WSL(2, R) group, which has also been proposed to obtain the BCH formulas [27].

3.2. SU(1, 1) AND SU(2) ALGEBRAS

The generators J_0, J_+, and J_- of the SU(1, 1) and SU(2) algebras satisfy the following commutation relations:

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-, \quad [J_-, J_+] = 2\sigma^2 J_0, \tag{19}$$

where $\sigma^2 = 1$ for the former and $\sigma^2 = -1$ for the latter, respectively.

There is a well-known realization for these algebras based on the Pauli matrices [1,10,16]. This 2×2 faithful matrix representation, which allows disentanglement of any exponential operator, can be easily derived by means of the procedure given before. For the sake of concreteness, we arbitrarily choose $X_1 = J_+$, $X_2 = J_0$, and $X_3 = J_-$. Since the operators

$$X_1 = \sigma(a^+)^2/2, \quad X_2 = (a^+a + 1/2)/2, \quad X_3 = \sigma a^2/2 \tag{20}$$

satisfy the commutation relations (19), we can choose $\mathcal{L}_Y$ to be the set of linear combinations of $Y_1 = a^+$ and $Y_2 = a$. Notice that $\mathcal{L}_Y$ is neither a subset of $\mathcal{L}_X$ nor even an algebra. According to eq. (8), the matrix for a general operator A of the form (2) belonging to the algebra is

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}a_2 & -a_1\sigma \\ a_3\sigma & -\frac{1}{2}a_2 \end{pmatrix}. \tag{21}$$

On arguing as before, we find that

$$\mathbf{O}_{\mathrm{II}}(\boldsymbol{\beta}) = \begin{pmatrix} \exp(\beta_2/2) - \sigma^2\beta_1\beta_3\exp(-\beta_2/2) & -\beta_1\sigma\exp(-\beta_2/2) \\ \beta_3\sigma\exp(-\beta_2/2) & \exp(-\beta_2/2) \end{pmatrix}, \tag{22}$$

and

$$\mathbf{O}_{\mathrm{I}}(\boldsymbol{\alpha}) = \begin{pmatrix} \cos(\delta) + \alpha_2\sin(\delta)/2\delta & -\alpha_1\sigma\sin(\delta)/\delta \\ \alpha_3\sigma\sin(\delta)/\delta & \cos(\delta) - \alpha_2\sin(\delta)/2\delta \end{pmatrix}, \tag{23}$$

where $\delta^2 = \sigma^2\alpha_1\alpha_3 - \alpha_2^2/4$. On equating these matrices, we obtain the BCH formulas

$$\begin{aligned} \beta_1 &= \alpha_1\sin(\delta)/[\delta\cos(\delta) - \alpha_2\sin(\delta)/2], \\ \beta_2 &= -2\ln[\cos(\delta) - \alpha_2\sin(\delta)/2\delta], \\ \beta_3 &= \alpha_3\sin(\delta)/[\delta\cos(\delta) - \alpha_2\sin(\delta)/2], \end{aligned} \tag{24}$$

which reduce to those derived before by means of the PDM for the special case $\alpha_1 = \tau$, $\alpha_2 = -i\alpha$, and $\alpha_3 = -\tau^*$, where * denotes the complex conjugation [13].

4. Further comments and conclusions

A matrix representation method has been developed which is useful in disentangling exponential operators. When it is faithful, the BCH formulas are obtained from a simple matrix equality; otherwise, it is found to be a useful complement of the PDM [13–15], giving directly the solutions to almost all the differential equations and avoiding the inversion of the matrix $\mathbf{M}$ when there is just one undetermined parameter. An attractive feature of the method is that $\mathbf{O}_{\mathrm{I}}$ and the factors in $\mathbf{O}_{\mathrm{II}}$ are easily obtained from the matrix for a single exponential of a general element of the Lie algebra (the operator A in eq. (2)).

When $\mathcal{L}_Y = \mathcal{L}_X$, the present matrix representation becomes the adjoint or regular one [16], in which case $m = n$. However, it is frequently possible to find a set $\mathcal{L}_Y$ with a smaller dimension.

Since the PDM has been applied to superalgebras and supergroups [14], it is reasonable to assume that a slightly modified version of the present procedure can also be useful in treating such cases.

In closing, it is worth mentioning that it is not always possible to write the exponential of an element of the Lie algebra as an arbitrarily ordered product. For instance, eqs. (24) are no longer valid when $2\delta\cos(\delta) = \alpha_2\sin(\delta)$, which can be satisfied by several values of the canonical coordinates of the first kind. The global validity of the BCH formulas has been discussed in a number of relevant papers [23].

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